

Enhancing synchronization in complex networks of coupled phase oscillators

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(Dated: February 2, 2008)

By a model of coupled phase oscillators, we show analytically how synchronization in *non-identical* complex networks can be enhanced by introducing a proper gradient into the couplings. It is found that, by pointing the gradient from the large-degree to the small-degree nodes on each link, increase of the gradient strength will bring forward the *onset* of network synchronization monotonically, and, with the same gradient strength, heterogeneous networks are more synchronizable than homogeneous networks. The findings are tested by extensive simulations and good agreement are found.

PACS numbers: 89.75.-k, 05.45.Xt

Synchronization in complex networks has been a topic of arising interest in recent years, mainly due to its implications to the practical processes observed in biological and neural systems [1, 2]. While most of the studies are focusing on the phenomenon of complete synchronization in networks of identical node dynamics [3], there are also interests in the collective behaviors in non-identical networks [4, 5]. The model of non-identical network is more representative to the realistic situations and, to analyze its dynamical properties, requires some special mathematical methods [4, 5]. Different to the studies in identical networks, in non-identical networks people are usually interested in the onset of the system coherence, i.e. the critical coupling from where the systems transits from the incoherent to coherent states [5, 6]. For general node dynamics, the critical coupling can be estimated by the method presented in Ref. [4], based on the information of the node dynamics and the largest eigenvalue of the adjacency matrix of the network. However, for the special case of coupled phase oscillators, the onset of network synchronization could be described more accurately by some other approaches. For instance, it has been shown that the critical coupling characterizing the onset can be efficiently predicted based on only the information of network degree distribution, i.e. the mean-field (MF) approach in Ref. [5]. In all these studies, the network couplings are considered as of uniform strength, i.e. the unweighted networks.

Noticing that couplings in realistic networks are usually directed and weighted and, in many cases, the direction and weight of the couplings are determined by a scalar field [7], it is thus natural to extend the study of non-identical network to the weighted case. For identical networks, it is shown that the synchronizability of a complex network can be significantly improved by introducing gradient into the couplings [8, 9, 10]. So far these findings are obtained from *identical networks* and referring to the transition of *global network synchronization*. In this paper, we are going to study the effects of coupling gradient on the *onset of system coherence in non-identical networks*, and explore their dependence to the network topology. The former study will extend the MF approach of Ref. [5]

to the situation of weighted network and the later stay could provide theoretical support to the findings of synchronization pathes in Ref. [11].

We consider network of N coupled phase oscillators of the following form (the generalized Kuramoto model [5])

$$\dot{\theta}_n = \omega_n + \varepsilon \sum_{m=1}^N C_{nm} \sin(\theta_m - \theta_n), \quad (1)$$

with θ_n and ω_n the phase and natural frequency of oscillator n respectively, ε is the overall coupling strength, and C_{nm} is an element of the coupling matrix C . In general, the matrix C is asymmetrical and the frequency ω_n follows some probability distribution $\rho(\omega)$. For the purpose of theoretical tractability, we assume that the network is densely connected and has a large size. Defining the global order parameter as $r \equiv \sum_{n=1}^N r_n / \sum_{n=1}^N d_n^{in}$, with $r_n e^{i\psi_n} \equiv \sum_{m=1}^N C_{nm} \langle e^{i\theta_m} \rangle_t$ the local order parameter and $d_n^{in} \equiv \sum_{m=1}^N C_{nm}$ the total *incoming* couplings of n , then the onset of the network synchronization is characterized by the critical coupling strength ε_c at which r starts to increase from 0. By the approaches of Ref. [5], we are able to obtain a similar equation for r in the region of $\varepsilon \geq \varepsilon_c$

$$r^2 = \frac{1}{\alpha_1 \alpha_2^2} \frac{\langle d^{in} d^{out} \rangle^3}{\langle (d^{in})^3 d^{out} \rangle \langle d^{in} \rangle^2} \left(\frac{\varepsilon}{\varepsilon_c} - 1 \right) \left(\frac{\varepsilon}{\varepsilon_c} \right)^{-3}, \quad (2)$$

with $d_n^{out} \equiv \sum_{m=1}^N C_{mn}$ is the total outgoing couplings departing from n , $\alpha_1 = 2/[\pi g(0)]$ and $\alpha_2 = -\pi g''(0)\alpha_1/16$ are two parameters determined by the first-order and second-order approximations of the frequency distribution $\rho(\omega)$, respectively. The critical coupling is given by the following equation

$$\varepsilon_c = \alpha \frac{\langle d^{in} \rangle}{\langle d^{in} d^{out} \rangle}, \quad (3)$$

with $\langle \dots \rangle$ denotes the system average. Please note that in our weighted model, the total incoming couplings d^{in} and the total

outgoing couplings d^{out} of each node are real and, in general, unequal. Our main task is to investigate how the distributions of d^{in} and d^{out} will affect the onset of network synchronization.

We start by considering an unweighted, symmetrical network described by adjacency $A = a_{nm}$, with $a_{nm} = 1$ if nodes n and m are connected, $a_{nm} = 0$ otherwise, and $a_{n,n} = 0$. The degree of node n is $k_n = \sum_{m=1}^N a_{nm}$. To introduce gradient into the couplings, we transform matrix A as follows. For each pair of connected nodes n and m in the network, we deduce an amount g from a_{nm} (the coupling that n receives from m) and add it to a_{mn} (the coupling that m receives from n). In doing this, the total couplings between n and m is keeping unchanged. Therefore a coupling gradient is generated which is pointing from node n to node m . Denoting the resulted matrix as S . The coupling matrix C is then defined as $C_{nm} \equiv k_n s_{nm} / \sum_{j=1}^N s_{nj}$ for the non-diagonal elements, and $C_{nn} = k_n$ for the diagonal elements. The coupling gradient from n to m thus is $\Delta C_{mn} = C_{mn} - C_{nm} = k_m s_{mn} / \sum_{j=1}^N s_{mj} - k_n s_{nm} / \sum_{j=1}^N s_{nj}$. In realistic systems, the direction and weight of each gradient are generally determined by a unified scalar field which, in the sense of network synchronization, is usually defined on the node degree [7, 8, 10]. Without losing generality, we make the gradient point from larger-degree to smaller-degree nodes on each link (the inverse case can be achieved by $g < 0$).

Now we discuss how the change of the gradient parameter g will affect the network synchronization. Noticing that in Eq. (3) the value of α is independent of g and, by the definition of C , we always have $\langle d^{in} \rangle = \langle k \rangle$, which is also independent of g . Therefore the introduction of gradient will only affect the value of $\langle d^{in} d^{out} \rangle$. Rearranging the node index by a descending order of their degrees, i.e. $k_1 > k_2 \dots > k_N$, then the outgoing couplings of n can be divided into two groups. Neighbors of node index $m < n$ have element $s_{mn} = 1 - g$ in matrix S and $C_{mn} < 1$ in matrix C (gradient points to n), while for nodes of index we have $s_{mn} = 1 + g$ in matrix S and $C_{mn} > 1$ in matrix C (gradient points to m). By this partition, the total outgoing couplings d_n^{out} is approximated as

$$d_n^{out} = k_n \left\{ \frac{1-g}{\Omega_i} P_{i<n} + \frac{1+g}{\Omega_i} P_{i>n} \right\}, \quad (4)$$

with $P_{i<n}$ ($P_{i>n}$) the probability for a randomly chosen node to have degree larger (smaller) than node n . $\Omega_i = \frac{1}{k_i} \sum_{j=1}^N s_{ij}$ is the normalizing factor defined on node. In calculating Ω_i , again, we can divide the neighbors of i into two groups. Nodes of index $j < i$ have $s_{ij} = 1 + g$ and nodes of index $j > i$ have $s_{ij} = 1 - g$. Based on this partition, we write

$$\Omega_i = 1 + g(P_{j<i} - P_{j>i}), \quad (5)$$

with P the same definition as that of Eq. (4). For heterogeneous networks of degree distribution $P(k) = Ck^{-\gamma}$ and, we have $\Omega_i = 1 + \frac{gC}{\gamma-1} \left\{ 2k_i^{1-\gamma} - k_{\max}^{1-\gamma} - k_{\min}^{1-\gamma} \right\}$, with k_{\max} and k_{\min} denote the largest and smallest node degrees of the network, respectively. Inserting this into Eq. (4), we obtain

$$d_n^{out} = k_n [F + G_n] \quad (6)$$

with

$$F = \frac{1}{2g} \{ (1+g) \ln(1+g) - (1-g) \ln(1-g) \} \quad (7)$$

and

$$G_n = -\ln \left[1 + g \frac{k_{\max}^{1-\gamma} + k_{\min}^{1-\gamma} - 2k_n^{1-\gamma}}{k_{\max}^{1-\gamma} - k_{\min}^{1-\gamma}} \right]. \quad (8)$$

Finally we have

$$\langle d^{in} d^{out} \rangle = \int_{k_{\min}}^{k_{\max}} [F + G(k)] k^2 P(k) dk \quad (9)$$

Eq. (9) is our main result which tells how the network synchronization (ε_c) changes with the coupling gradient (g) and the network topology (γ).

From Eq. (6) we know that, in comparison with the unweighted networks, the introduction of the coupling gradient changes only the weight $H \equiv F + G$ of the outgoing couplings on each node, while in this process the total strength of the outgoing couplings is keeping unchanged. That is to say, gradient makes the distribution of H change from an even form ($H \equiv 1$ in unweighted network) to an uneven form ($H = H(g, k)$ in weighted network). Physically, the term F can be understood as a summation of the symmetrical part of the couplings on each node, i.e. $F \sim \sum_{i=1}^N \min(C_{ni}, C_{in})$, which only depends on parameter g and will be decreased as g is increased. In contrast, the term G is a joint function of g and k_n . While G increases with g , its exact value, however, are strongly modified by the node degree: larger degree assumes larger G [Eq. (8)]. The joint effect of F and G will divide the nodes into two groups. Nodes of degrees larger than some critical value k_c have weight $H > 1$, while nodes of degrees smaller than k_c have weight $H < 1$. The critical degree k_c can be calculated from the equation of $H = 1$. Under the assumption of $k_{\max} \gg k_{\min}$, we have

$$k_c = \ln \left[\frac{1}{2} - \frac{1}{2g} (1 - e^{F-1}) \right]^{\frac{1}{1-\gamma}} k_{\min}. \quad (10)$$

The uneven distribution of H can be further understood by considering its approximations at $k \approx k_{\max}$ and $k \approx k_{\min}$, which results in $H_{k \approx k_{\max}} = \frac{1}{2g} (1+g) \ln \frac{1+g}{1-g}$ and $H_{k \approx k_{\min}} = \frac{1}{2g} (1-g) \ln \frac{1+g}{1-g}$. Clearly, we have $H_{k \approx k_{\max}} > H_{k \approx k_{\min}}$. Since the sum of H over the network is fixed, i.e. $\sum_{i=1}^N H_i = N$, the gradient effect thus can be roughly regarded as a shifting of weight H from smaller-degree to higher-degree nodes.

For scale-free networks generated by the standard BA growth model [1], we have $k_{\max} \approx k_{\min} N^{\frac{1}{\gamma-1}}$. Inserting this relation into Eq. (8) we obtain

$$H = F - \ln \left[1 - \beta + 2\beta \left(\frac{k}{k_{\min}} \right)^{1-\gamma} \right], \quad (11)$$

which basically tells the following: for fixed gradient parameter g , increasing the homogeneity of the network, i.e. increasing exponent γ , will make the distribution of H more homogeneous and, as a result, the network synchronization will be

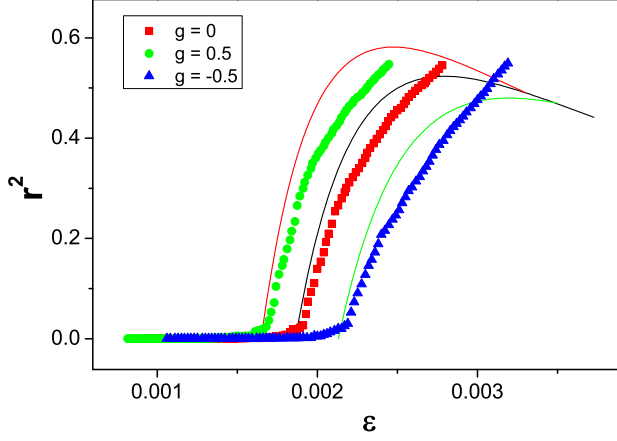


FIG. 1: (Color online) For scale-free network of 1500 nodes, average degree 400, and degree exponent $\gamma = 3$, the variation of the squared order parameter r^2 as a function of the coupling strength ε in the region of $\varepsilon \in [0.5\varepsilon_c, 1.5\varepsilon_c]$ by using gradient parameters $g = 0.5$ (the left symbol curve), $g = 0$ (the middle symbol curve), and $g = -0.5$ (the right symbol curve). Apparently, the onset point of synchronization is shifted to the small values as g increases. Each data is averaged over 10 network realizations. The three line curves are plotted according to Eq. (2), which predicts the behavior of r^2 reasonably well in the region of $\varepsilon \in [\varepsilon_c, 0.3\varepsilon_c]$. In all the three cases, the numerical results of the critical couplings ε_c are in good agreements with the theoretical predictions calculated from Eq. (3).

suppressed (i.e. the value of ε_c will increase with γ). Eq. (11) gives the dependence of network synchronization on network topology.

Now the effect of coupling gradient and the effect of topology on the starting of synchronization in nonidentical networks can be summarized as follows. The changes of the gradient strength g or the degree exponent γ do not change the total coupling cost of the network, they will only redistribute the weights of the outgoing couplings at each node according to its degree information. By adding gradient, the outgoing couplings at the small-degree nodes (of degree $k < k_c$) will be reduced by an amount and added to those of large-degree nodes (of degree $k > k_c$). This will induce a heterogeneous distribution in H which in turn will decrease the threshold coupling ε_c (see Eq. (3)). This enhancement of network synchronization, however, is modulated by the network topology. By increasing the degree exponent γ , the distribution of H tends to be homogeneous (i.e. $H \sim 1$) and, consequently, network synchronization is suppressed. These are the mechanisms governing the effects of gradient and topology on network synchronization. The above analysis shows that: 1) the synchronization of non-identical networks can be enhanced by coupling gradient; and 2) in comparison with homogeneous networks, heterogeneous networks take more advantages from the coupling gradient.

We now provide the numerical results. The networks are generated by a generalized BA model [12], which is able to generate networks of varying degree exponent γ . The fre-

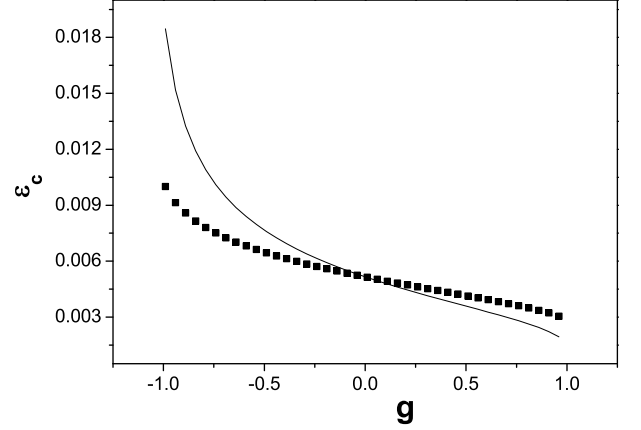


FIG. 2: For a scale-free network of 5000 nodes, average degree 100, and degree exponent $\gamma = 3$, the variation of the critical coupling strength ε_c as a function of the gradient parameter g . The solid line represents the theoretical results predicted by Eq. (4).

quency distribution is given by $\rho(\omega) = (3/4)(1 - \omega^2)$ for $-1 < \omega < 1$ and $\rho(\omega) = 0$ otherwise. The initial phase θ of each oscillator is randomly chosen within range $[0, 2\pi]$. A transition time $T = 100$ is discarded, and the value of r^2 is calculated over another period of $T = 100$. To show the gradient effects on network synchronization, we have calculated the variations of the squared order parameter r^2 as a function of the coupling strength ε for three different gradient parameters: $g = 0, 0.5$ and -0.5 . (According to our definition, $g < 0$ means that gradient is pointing from smaller to larger nodes.) The results are plotted in Fig. 1. Clearly, the critical coupling strength ε_c is shifted to small values as g is increased. The three lines plotted in Fig. 1 represents the theoretical results of Eq. (2), which fit well with the numerical results in the neighboring region of the onset. More importantly, the position of the onset coupling ε_c is predicted precisely by Eq. (3). (The precision of this predication is dependent on the size and connectivity of the network, larger and denser networks give better results.)

To have a global picture on the gradient effect, we plot Fig. 2 the simulation result of the variation of ε_c as a function of g . It is shown that, as g changes from -1 to 1 , the value of ε_c is *monotonically* decreased. This process of synchronization enhancement is well captured by Eq. (8), especially in the region of $g > 0$. Since in our analysis we have assumed the network to be of very large size and of dense connectivity, the mismatch between the theoretical and numerical results in Fig. 2 is reasonable.

Simulations have been also conducted on the dependence of ε_c on γ . By the generalized BA model [12], we vary the degree exponent γ continuously from 3 to 25, while keeping the size and average degree of the network unchanged. As we have predicted [Eq. (11)], in Fig. 3 it is found that, for each value of g , the critical coupling ε_c will increase monotonically with the degree exponent γ . Specially, for the case

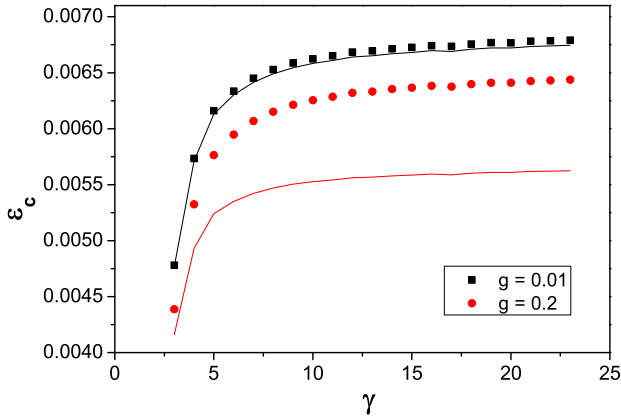


FIG. 3: (Color online) For a scale-free network of 5000 nodes, average degree 100, the variation of the critical coupling strength ε_c as a function of the degree exponent γ under the gradient parameters $g = 1 \times 10^{-2}$ (the upper symbol curve) and $g = 0.2$ (the lower symbol curve). For both cases, ε_c increases with γ . The solid lines are the theoretical results predicted by Eq. (11).

of $g = 1 \times 10^{-2}$ in Fig. 3, the numerical results are in good agreements with the theoretical results of Eq. (10). As g increases the mismatch between the theoretical and numerical results is enlarged, especially for networks of larger γ . Again, by increasing size and coupling density of the network, the mismatch can be alleviated.

A few remarks are in order. Firstly, while our theory gives well approximations on the collective behavior in densely connected large networks, our findings about gradient effects of their dependence to network topologies are general for any

network. The amazing thing is that, for network of given degree distribution (not limited to the scale-free type), our theory tells how much improvement could the network benefits from a given gradient. From the findings, we are able to not only point out clearly the optimal configuration for synchronization, which happens when $g = 1$ [Fig. 2], but also have a systematic understanding on the *transition* from unweighted to optimal network, and, more importantly, the underlying mechanisms that govern this transition. Secondly, although similar findings about the gradient effects had been discovered previously in the study of global synchronization of identical networks [8, 9, 10], our analyses, however, are focusing on the *onset synchronization in non-identical networks*. Another difference is, by adopting the generalized Kuramoto model, that we are able to show *analytically* how the coupling gradient affects synchronization (see Eq. (9) and Fig. 2) and what is the role of network topology in this process ((see Eq. (11) and Fig. 3). It is noticed that in Ref. [[11]] the authors found numerically that the onset of synchronization in scale-free networks happens in advance to that of homogeneous networks, which, according to the approximation of Eq. (11), can be easily understood.

In summary, we have studied the effects of coupling gradient on the onset of synchronization in nonidentical complex networks. It is found that: 1) network synchronization can be enhanced by introducing gradient into to the couplings; and 2) in terms of the onset of synchronization, heterogeneous networks are more synchronizable than homogeneous networks. We hope these findings to be helpful in understanding the collective behaviors in realistic systems.

YCL was also supported by AFOSR under Grants No. FA9550-06-1-0024 and No. FA9550-07-1-0045.

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Synchronization of complex networks has received many interests in recent years, mainly due to its important implications to the processes in biological and neural systems [1, 2]. While most of the studies are focusing on the phenomenon of complete synchronization in identical networks (the same dynamics for all nodes) [3], there are also interests on the collective behaviors of non-identical networks (node dynamics is non-identical) [4, 5]. Comparing to identical networks, non-identical networks are more representative of the realistic situations and, to analyze their dynamical properties, a set of special mathematical treatments had been developed [4, 5]. Meanwhile, in non-identical networks people are more interested with the onset of network synchronization, i.e. the critical coupling where the system transits from incoherent to coherent states [5, 6], instead of global synchronization studied in identical networks. In general, the onset coupling strength of a non-identical network can be estimated by the method proposed in Ref. [4], based on the information of node dynamics and network topology. However, for coupled phase oscillators, this estimation can be greatly simplified and improved by some special approaches. For instance, by the mean-field (MF) approach of Ref. [5], the onset coupling strength can be predicted by only the information of degree distribution. In all these studies, the network couplings are considered as having the uniform strength, i.e. the networks are unweighted.

Noticing that couplings in realistic networks are usually directed and weighted and, in many cases, the direction and weight of each coupling are determined by a scalar field of the network [7], it is natural to extend the studies of non-identical networks to the weighted cases. For identical network, it has been shown that the synchronizability of a network can be significantly improved by introducing some proper gradients into the couplings [8, 9, 10]. So far these findings are obtained from *identical networks* and referring to the transition of *global network synchronization*. In this paper, we are going to study the effects of coupling gradient on the *onset of system coherence in non-identical networks*, and explore their dependence to the network topology. The former study will extend

the MF approach of Ref. [5] to the situation of weighted networks, and the later study could provide insights to the problem of synchronization pathes observed in Ref. [11].

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$$r^2 = \frac{1}{\alpha_1 \alpha_2^2} \frac{\langle d^{in} d^{out} \rangle^3}{\langle (d^{in})^3 d^{out} \rangle \langle d^{in} \rangle^2} \left(\frac{\varepsilon}{\varepsilon_c} - 1 \right) \left(\frac{\varepsilon}{\varepsilon_c} \right)^{-3}. \quad (2)$$

In Eq. (2), $d_n^{out} = \sum_{m=1}^N c_{mn}$ denotes the total outgoing coupling strength of node n , $\alpha_1 = 2/[\pi\rho(0)]$ and $\alpha_2 = -\pi\rho''(0)\alpha_1/16$ are two parameters determined by the first-order ($\rho(0)$) and second-order ($\rho''(0)$) approximations of the frequency distribution, respectively. The critical coupling ε_c reads

$$\varepsilon_c = \alpha_1 \frac{\langle d^{in} \rangle}{\langle d^{in} d^{out} \rangle}, \quad (3)$$

with $\langle \dots \rangle$ denotes a system average. Please note that in our weighted model, the total incoming coupling strength d_n^{in} and

the total outgoing coupling strength d^{out} at each node are real values and, in general, are unequal. The main task of this paper is to investigate how the distributions of d^{in} and d^{out} will affect the network synchronization.

We start by considering an unweighted, symmetrical network described by adjacency $A = \{a_{nm}\}$, with $a_{nm} = 1$ if nodes n and m are connected, $a_{nm} = 0$ otherwise, and $a_{n,n} = 0$. The degree of node n is $k_n = \sum_{m=1}^N a_{nm}$. To introduce the coupling gradient, we transform the adjacency matrix A as follows. For each pair of connected nodes n and m in the network, we deduce an amount g from a_{nm} (the coupling that n receives from m) and add it to a_{mn} (the coupling that m receives from n). In doing this, the total coupling strength between n and m is keeping unchanged. However, a coupling gradient will be generated, which is pointing from n to m . Denoting the resulted matrix as S . The coupling matrix C is then defined as $c_{nm} = k_n s_{nm} / \sum_{j=1}^N s_{nj}$. The coupling gradient from n to m therefore is $\Delta c_{mn} = c_{mn} - c_{nm} = k_m s_{mn} / \sum_{j=1}^N s_{mj} - k_n s_{nm} / \sum_{j=1}^N s_{nj}$. In practice, the direction of the coupling gradient is determined by a scalar field which, in terms of synchronization, is usually adopted as the node degree [7, 8, 10]. Without losing the generality, we arrange coupling gradient flow from the larger-degree to the smaller-degree nodes on each link (the inverse situation can be achieved by setting $g < 0$).

Now we discuss how the change of the gradient parameter g will affect the network synchronization. Noticing that in Eq. (3) the value of α_1 is independent of g and, in constructing matrix C , we have $\langle d^{in} \rangle = \langle k \rangle$ which is also independent of g . Therefore the introduction of gradient will only affect the value of $\langle d^{in} d^{out} \rangle$ in Eq. (3). Rearranging node index by a descending order of node degree, i.e. $k_1 > k_2 \dots > k_N$, then the outgoing couplings of node n can be divided into two groups. Neighbors of node index $m < n$ have elements $s_{mn} = 1 - g$ in matrix S and $c_{mn} < 1$ in matrix C (coupling gradient points to n), while for nodes of index $m > n$ we have $s_{mn} = 1 + g$ in matrix S and $c_{mn} > 1$ in matrix C (coupling gradient departs from n). By this partition, the total outgoing coupling strength d_n^{out} of node n reads

$$d_n^{out} = k_n \left\{ \frac{1-g}{\Omega_i} P_{i<n} + \frac{1+g}{\Omega_i} P_{i>n} \right\}, \quad (4)$$

with $P_{i<n}$ ($P_{i>n}$) is the probability for a randomly chosen node to have degree larger (smaller) than that of node n , and $\Omega_i = \frac{1}{k_i} \sum_{j=1}^N s_{ij}$ is the normalizing factor defined on node. In calculating Ω_i , again, we can divide the neighbors of i into two groups. Nodes of index $j < i$ have $s_{ij} = 1 + g$ and nodes of index $j > i$ have $s_{ij} = 1 - g$. Based on this partition, we obtain

$$\Omega_i = 1 + g(P_{j<i} - P_{j>i}), \quad (5)$$

with P the same meaning as that of Eq. (4). For heterogeneous networks of degree distribution $P(k) = Ck^{-\gamma}$, we have $\Omega_i = 1 + \frac{gC}{\gamma-1} \left\{ 2k_i^{1-\gamma} - k_{\max}^{1-\gamma} - k_{\min}^{1-\gamma} \right\}$, with k_{\max} and k_{\min} denote the largest and smallest node degrees of the network, respectively. Inserting this into Eq. (4), we obtain

$$d_n^{out} = k_n [F + G_n] \quad (6)$$

with

$$F = \frac{1}{2g} \{ (1+g) \ln(1+g) - (1-g) \ln(1-g) \} \quad (7)$$

and

$$G_n = -\ln \left[1 + g \frac{k_{\max}^{1-\gamma} + k_{\min}^{1-\gamma} - 2k_n^{1-\gamma}}{k_{\max}^{1-\gamma} - k_{\min}^{1-\gamma}} \right]. \quad (8)$$

Finally we have

$$\langle d^{in} d^{out} \rangle = \int_{k_{\min}}^{k_{\max}} [F + G(k)] k^2 P(k) dk \quad (9)$$

Eq. (9) is our main result which tells how the network synchronization (ε_c) changes with the coupling gradient (g) and the network topology (γ).

From Eq. (6) we know that the introduction of coupling gradient changes only the weights $H = F + G$ of the outgoing couplings at each node, while the total coupling cost of the network is keeping unchanged. That is to say, gradient changes the distribution of H from an even form ($H = 1$ in unweighted network) to an uneven form ($H = H(g, k)$ in weighted network). Physically, we can regard F as the symmetrical part of the couplings on each link, i.e. $F \sim \min(c_{ni}, c_{in})$, which depends only on the gradient parameter g and is decreased as g increases. In contrast, the term G is a joint function of g and k_n . While G increases with g monotonically, its exact value, however, are strongly modified by the node degree: node of larger degree assumes larger G [Eq. (8)]. According to the value of H , we are able to divide the network nodes into two groups. Nodes which have degree larger than k_c have $H > 1$, while those have degree smaller than k_c have $H < 1$. The critical degree k_c can be calculated by requiring $H = 1$. Under the assumption of $k_{\max} \gg k_{\min}$, we get

$$k_c = \ln \left[\frac{1}{2} - \frac{1}{2g} (1 - e^{F-1}) \right]^{\frac{1}{1-\gamma}} k_{\min}. \quad (10)$$

The uneven distribution of H becomes even clear when considering the extreme cases of $k \approx k_{\max}$ and $k \approx k_{\min}$. From Eq. (10) we have $H_{k \approx k_{\max}} = \frac{1}{2g} (1+g) \ln \frac{1+g}{1-g}$ and $H_{k \approx k_{\min}} = \frac{1}{2g} (1-g) \ln \frac{1+g}{1-g}$. Clearly, $H_{k \approx k_{\max}} > H_{k \approx k_{\min}}$. Since the sum of H is constant for the network, i.e. $\sum_{i=1}^N H_i = N$, the gradient effect thus can be understood as a shifting of partial of the weight H from smaller-degree to higher-degree nodes.

For scale-free networks generated by the standard BA growth model [1], we have $k_{\max} \approx k_{\min} N^{\frac{1}{\gamma-1}}$. Inserting this into Eq. (8) we obtain

$$H = F - \ln \left[1 - g + 2g \left(\frac{k}{k_{\min}} \right)^{1-\gamma} \right], \quad (11)$$

which basically tells the following: for a fixed gradient strength g , increase of the network homogeneity, i.e. the degree exponent γ , will make the distribution of H more homogeneous and, as a result, suppress the network synchronization. In other words, the value of ε_c increases with the increase of γ .

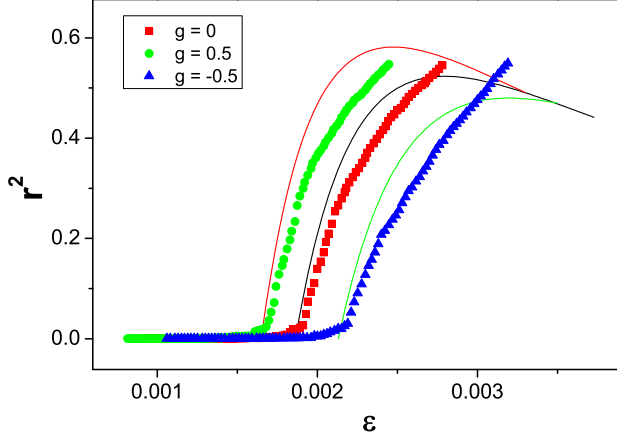


FIG. 1: (Color online) For scale-free network of 1500 nodes, average degree 400, and degree exponent $\gamma = 3$, the variation of the squared order parameter r^2 as a function of the coupling strength ε in the region of $\varepsilon \in [0.5\varepsilon_c, 1.5\varepsilon_c]$ under gradient strengths $g = 0.5$ (the left symbol curve), $g = 0$ (the middle symbol curve), and $g = -0.5$ (the right symbol curve). Apparently, the onset of synchronization is shifted to small coupling strengths at larger values of g . Each data is averaged over 10 network realizations. The three line curves are plotted according to Eq. (2), which predicts the behavior of r^2 reasonably well in the region of $\varepsilon \in [\varepsilon_c, 0.3\varepsilon_c]$. Specially, the numerical results of the critical coupling strength ε_c are in good agreements with the theoretical results predicted by Eq. (3).

The effects of coupling gradient and network topology on network synchronization can be summarized as follows. The change of the gradient strength g or the degree exponent γ does not change the total coupling cost of the network, it only redistributes the weight of the outgoing couplings at each node according to its degree information. When gradient $g > 0$ is introduced, the outgoing couplings of small-degree nodes (having degree $k < k_c$) will be reduced by an amount and added to those of large-degree nodes (having degree $k > k_c$). This will induce a heterogeneous distribution in H which in turn will decrease the value of ε_c (see Eq. (3)). This enhancement of network synchronization, however, is modulated by the network topology. By increasing the degree exponent γ , the distribution of H tends to be homogeneous (i.e. $H \sim 1$) and, consequently, the network synchronization is suppressed. These are the mechanisms that govern the functions of coupling gradient and network topology. Therefore our findings are: 1) by coupling gradient, synchronization in non-identical networks can be enhanced; and 2) in comparison with homogeneous networks, heterogeneous networks take more advantages from coupling gradient and, under the same gradient strength, are more synchronizable.

We now provide the numerical results. The networks are generated by a generalized BA model [12], which is able to generate networks of varying degree exponent γ . The frequency distribution is given by $\rho(\omega) = (3/4)(1 - \omega^2)$ for $-1 < \omega < 1$ and $\rho(\omega) = 0$ otherwise. The initial phase θ of each oscillator is randomly chosen within range $[0, 2\pi]$.

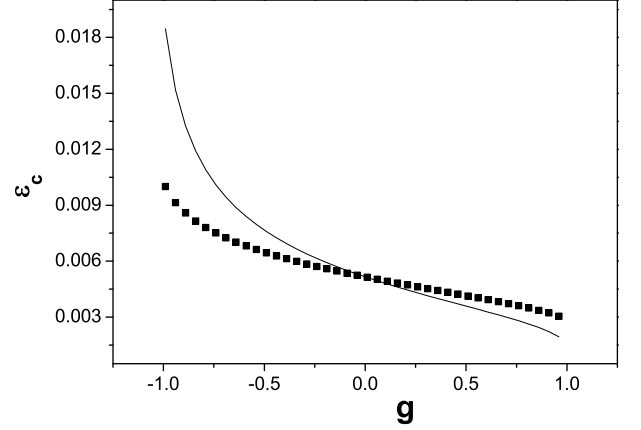


FIG. 2: For a scale-free network of 5000 nodes, average degree 100, and degree exponent $\gamma = 3$, the variation of the critical coupling ε_c as a function of the gradient parameter g . The solid line represents the theoretical results predicted by Eq. (4).

A transition time $T = 100$ is discarded, and the value of r^2 is calculated over another period of $T = 100$. To show the gradient effects on network synchronization, we have first calculated the variations of the squared order parameter r^2 as a function of the coupling strength ε under three gradient parameters: $g = 0, 0.5$ and -0.5 . (According to our definition, $g < 0$ means that coupling gradient is pointing from small-degree to larger-degree nodes.) The results are plotted in Fig. 1. Clearly, ε_c becomes smaller at larger g . The three lines in Fig. 1 are plotted by Eq. (2), which fit well with the numerical results in the neighboring region of the onset point. More importantly, the value of ε_c is predicted precisely by Eq. (3). (The precision of the predications dependent on the system size and the coupling density, larger and denser networks give better results.)

To have a global picture on the gradient effect, we plot Fig. 2 the variation of the critical coupling strength ε_c as a function of the gradient parameter g . It is shown that, as g changes from -1 to 1 , the value of ε_c is *monotonically* decreased. This process of synchronization enhancement is well captured by Eq. (8) (the line curve in Fig. 2), especially when $g \approx 0$. Since in our analysis we have assumed the network to be very large and dense, the mismatch between the theoretical and numerical results in Fig. 2 is understandable.

Simulations have been also conducted on the dependence of ε_c on γ . By the generalized BA model [12], we vary the degree exponent γ continuously from 3 to 25, while keeping the size and average degree unchanged. As the prediction of Eq. (11), it is found that, for each value of g , the critical coupling strength ε_c increases monotonically with the increase of the degree exponent γ [Fig. 3]. Specially, for the case of $g = 1 \times 10^{-2}$, the numerical results are in good agreements with the theoretical approximations of Eq. (10). As g increases, the mismatch between theoretical and numerical results will be enlarged which, again, can be alleviated by in-

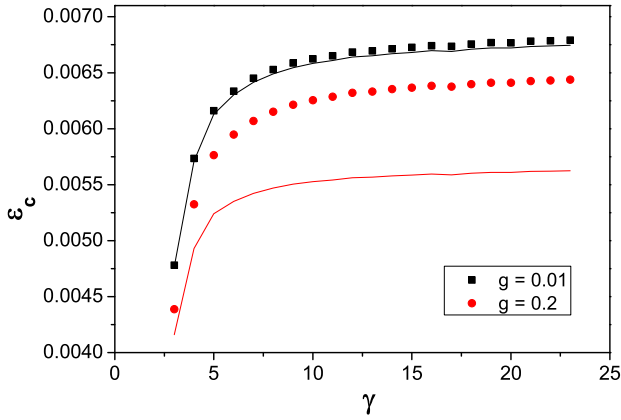


FIG. 3: (Color online) For a scale-free network of 5000 nodes, average degree 100, the variation of the critical coupling strength ε_c as a function of the degree exponent γ under gradient parameters $g = 1 \times 10^{-2}$ (the upper symbol curve) and $g = 0.2$ (the lower symbol curve). In both cases, ε_c increases with γ . The solid lines are the theoretical results predicted by Eq. (11).

creasing the network size and density.

A few remarks are in order. Firstly, while our theory is derived from large and densely connected networks, the general finding that synchronization can be enhanced by gradient couplings applies to any network. The interesting thing is that, for a network of given degree distribution (not limited to the scale-free type), our theory predicts quantitatively

how large an improvement the network could benefit for a given gradient. From the findings, we are able to not only point out the optimal configuration for network synchronization, which happens at $g = 1$ [as shown in Fig. 2], but also have a systematic understanding on the *transition* of the system performance as a function of gradient, and, more importantly, the underlying mechanisms that govern this transition. Secondly, although similar findings about gradient effects had been discovered previously in studying global synchronization in identical networks [8, 9, 10], our analyses, however, are focusing on the *onset of synchronization in non-identical networks*. Another difference is that, by adopting the generalized Kuramoto model, we are able to show *analytically* how the change of the coupling gradient will affect the synchronization (Eq. (9) and Fig. 2) and what is the role of network topology in this process (Eq. (11) and Fig. 3). It is noticed that in Ref. [11] the authors found numerically that the onset of synchronization in scale-free networks happens in advance to that of homogeneous networks, which can be understood readily from Eq. (11).

In summary, we have studied the effects of coupling gradient on the onset of synchronization in nonidentical complex networks and found that, by coupling gradient, the network synchronization can be significantly enhanced and, in comparison with homogeneous networks, heterogeneous networks are more synchronizable. We hope these findings could give insights to the collective behaviors in realistic systems.

YCL was also supported by AFOSR under Grants No. FA9550-06-1-0024 and No. FA9550-07-1-0045.

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